Numerical approaches to analysis

Using numerical solutions to problems
The Solver in Excel

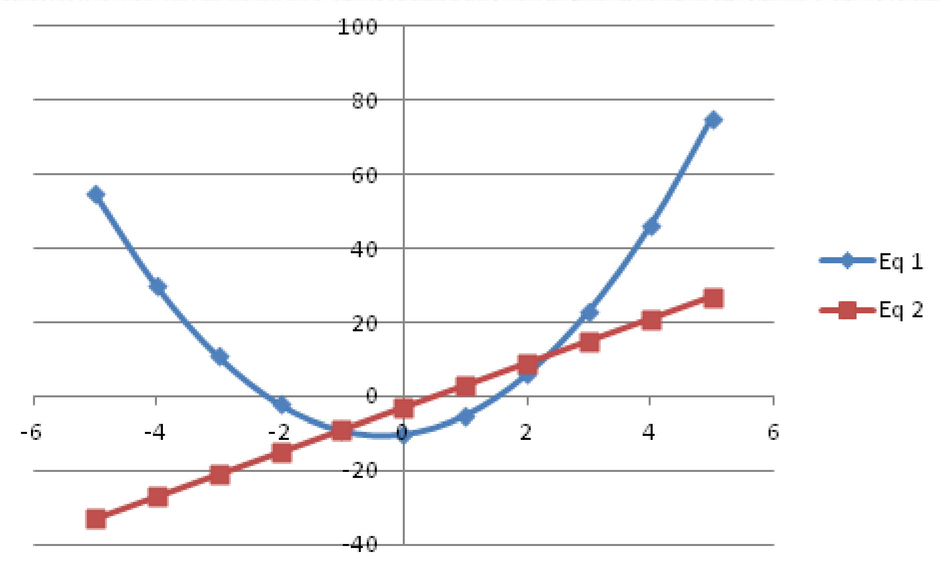
Analytical vs. numerical solutions

- Analytical solutions = solving for the term of interest in an equation
- Numerical solutions = approximate solutions found with iterative, trial and error methods
- Numerical solutions are approximate, but if the approximation is good enough it doesn’t matter
- Example: solutions to a system of equations

Solutions to equations

- Two equations, one linear and one quadratic
  - $y = 3x^2 + 2x - 10$
  - $y = 6x - 3$
- Roots of the equations: where do the lines cross?
  - That is, what values of x have the same y for both equations?
- First, graph them – how many solutions should we expect?
- There is an analytical solution to this

How many solutions places do the line cross? Plickers...



The analytical solution

$$y = 3x^2 + 2x - 10$$
$$y = 6x - 3$$

$$6x - 3 = 3x^2 + 2x - 10$$

$$0 = 3x^2 - 4x - 7$$

Quadratic equation in standard form, find roots with quadratic formula →

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{-(-4) + \sqrt{(-4)^2 - 4(3)(-7)}}{2(3)}$$

$$x = 2\frac{1}{3}, \quad y = 11$$

Plug x into either equation to get y

$$x = -1, \quad y = -9$$

$$\frac{-(-4) - \sqrt{(-4)^2 - 4(3)(-7)}}{2(3)}$$

Numerical approach

- Re-write the equations as: $0=3x^2+2x-10-y$

$$0=6x-3-y$$

- Set values for X and Y to be the same in both equations
  - Initial guesses will make the equations not equal 0 at first
  - Change the values of X and Y until both equations equal 0
- We'll get solutions for both X and Y at the same time

Numerical method – in Excel

	A	B	C
1	F1	3x^2+2x -10 -y	
2	F2	6x - 3 - y	
3			
4	x	0	
5	y	0	
6			
7	F1	-10	
8	F2	-3	
9			
10	Sum sq	109	
11			

The initial setup...

1. Enter initial guesses for x and y

2. Calculate F1 and F2 using x and y

3. Square F1 and F2 and sum them

Try new set of numbers for x and y

	A	B	C
1	F1	3x^2+2x -10 -y	
2	F2	6x - 3 - y	
3			
4	x	1	
5	y	1	
6			
7	F1	-6	
8	F2	2	
9			
10	Sum sq	40	
11			

1. Change the values of x and y
2. Calculate F1 and F2 using new values for x and y
3. Sum of squared values gets closer to 0 – moving in the right direction!

Not 0 yet...

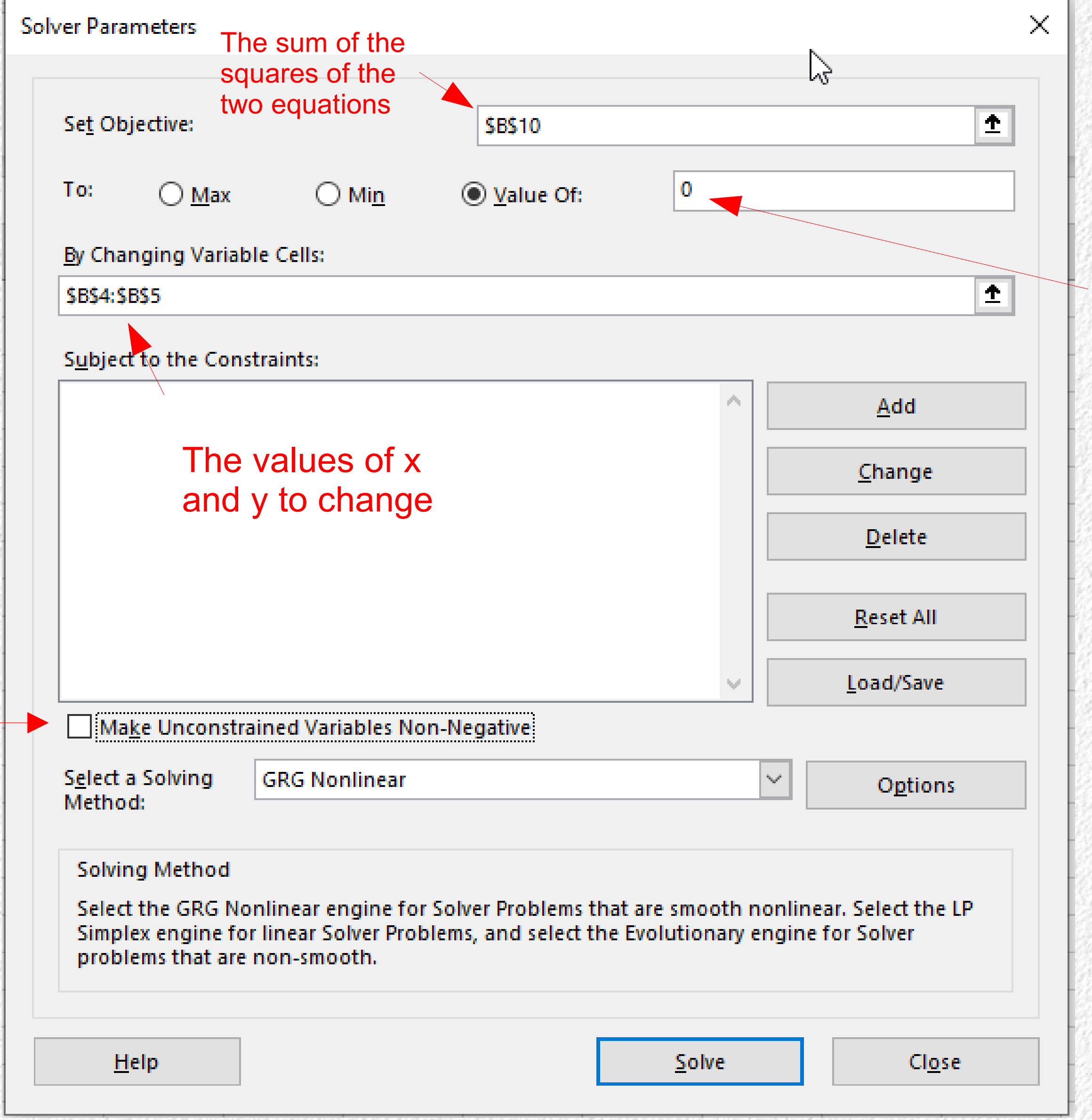
- Better
- Blind search could take a really long time
- There are good search algorithms that converge on solutions quickly
- Excel's Solver uses these

	A	B	C
1	F1	3x^2+2x -10 -y	
2	F2	6x - 3 - y	
3			
4	x	2	
5	y	10	
6			
7	F1	-4	
8	F2	-1	
9			
10	Sum sq	17	
11			

Optimization algorithms used by Excel's Solver

- Excel picks a method to use based on the formulas in the spreadsheet
  - For linear problems, uses the Simplex method
  - For non-linear problems it uses a generalized gradient method
- Both require initial guesses of the solutions
- Both can accept constraints (i.e. only positive values considered)
- Both are iterative (i.e. new values chosen until no more improvement at the level of precision desired)

Solver



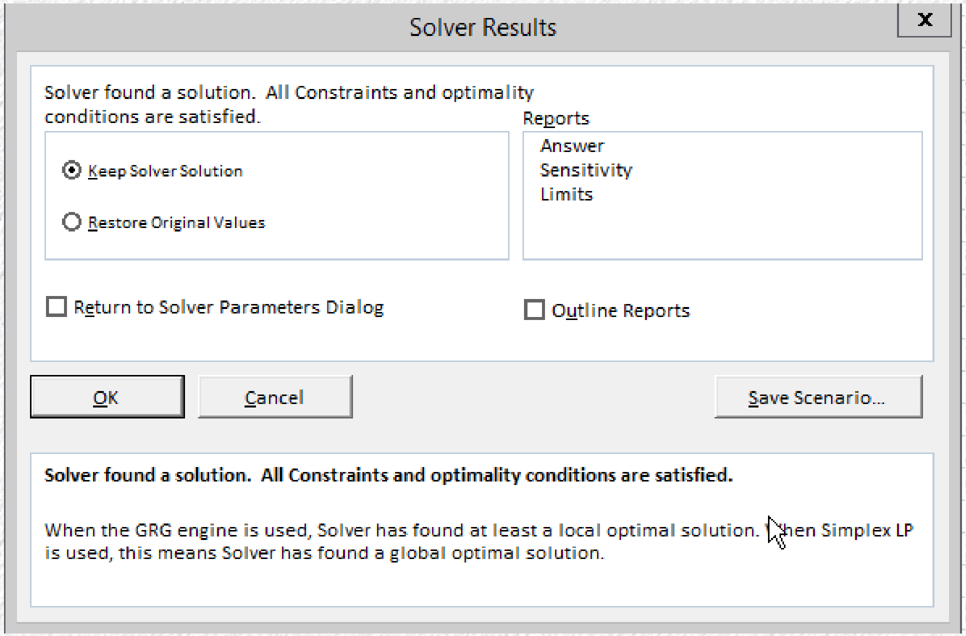
The value we want the sum in B10 to be (could also ask for min)

Allow x and y to be negative

	A	B	C
1	F1	3x^2+2x -10 -y	
2	F2	6x - 3 - y	
3			
4	x	0	
5	y	0	
6			
7	F1	-10	
8	F2	-3	
9			
10	Sum sq	109	
11			

The first solution

	A	B	C
1	F1	3x^2+2x -10 -y	
2	F2	6x - 3 - y	
3			
4	x	2.333368	
5	y	11.00036	
6			
7	F1	0.000189	
8	F2	-0.00015	
9			
10	Sum sq	5.95E-08	

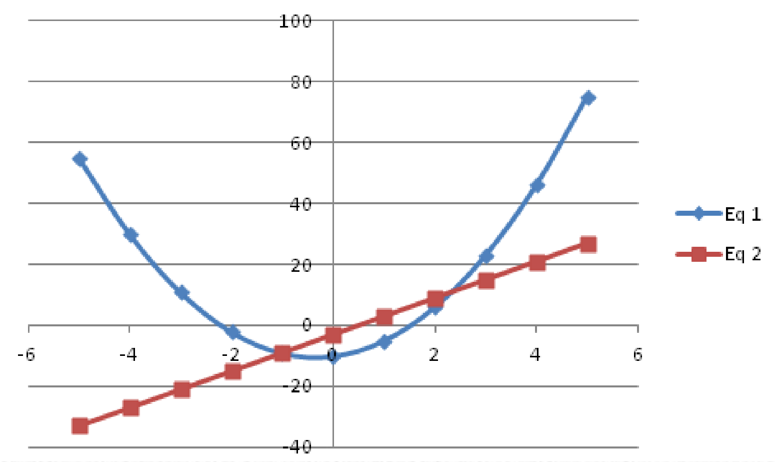


$$x = 2\frac{1}{3}, \quad y = 11$$

← *Do they match? Plickers...*

What about the second solution?

- There are two points of intersection of the lines → two solutions
- Our trial and error approach gave us one, now we need the other
- To get the second, try another starting point closer to the other solution and run Solver again



Start closer to the second solution...

	A	B	C
1	F1	3x^2+2x -10 -y	
2	F2	6x - 3 - y	
3			
4	x	-1	
5	y	-10	
6			
7	F1	1	
8	F2	1	
9			
10	Sum sq	2	
11			

Second solution

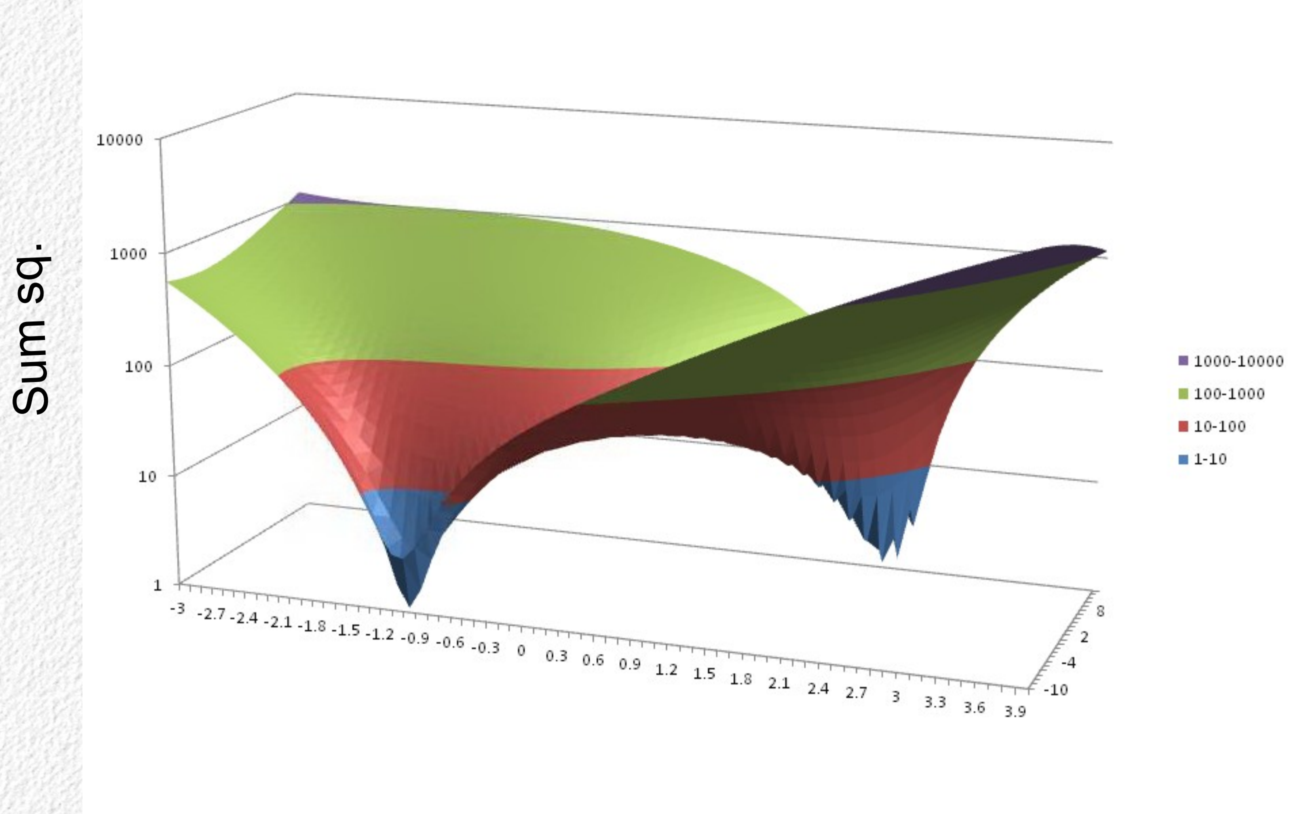
	A	B	C
1	F1	3x^2+2x -10 -y	
2	F2	6x - 3 - y	
3			
4	x	-1	
5	y	-8.99996	
6			
7	F1	-5.6E-05	
8	F2	-1.4E-05	
9			
10	Sum sq	3.32E-09	
11			

Note that this works best when you know how many solutions to expect – graphing is a very important first step

If you don't know, choose several different starting positions to increase the chance you'll find the solutions

$$x = -1, \quad y = -9$$

Sum of squared functional values at different x,y values



The slope of the surface determines which solution you'll find

Usually, whichever you start closer to is the one you'll find

Numerical methods in biology

- Some equations can't be solved analytically, have to use numeric methods
- Example: life tables
  - Data on age-specific birth and death rates for populations
- The basic data are:
  - The number of individuals alive each year, starting from birth until all are dead = n_x
  - The number of female offspring per females of age x is b_x

Life table for a squirrel population

	A	B	C
1	Age	n(x)	b(x)
2	0	1000	0
3	1	458	1.28
4	2	352	2.28
5	3	229	2.28
6	4	154	2.28
7	5	99	2.28
8	6	87	2.28

Euler's equation

- Population growth rate is the balance between birth and death rate,
  r = birth rate – death rate
- If r is positive, the population is growing
- The best estimate of r from a life table is the value that satisfies Euler's equation:

$$1=\sum l_x b_x e^{-rx}$$

- x (age), l_x, b_x are all known, e is a constant
- Equation can't be solved analytically, but we can find r numerically with the Solver

Convert number alive (n_x) to proportion alive (l_x)

	A	B	C	D
1	Age	n(x)	b(x)	l(x)
2	0	1000	0	1
3	1	458	1.28	0.458
4	2	352	2.28	0.352
5	3	229	2.28	0.229
6	4	154	2.28	0.154
7	5	99	2.28	0.099
8	6	87	2.28	0.087

Multiply proportion alive by birth rate ($l_x b_x$)

	A	B	C	D	E
1	Age	n(x)	b(x)	l(x)	l(x)b(x)
2	0	1000	0	1	0
3	1	458	1.28	0.458	0.58624
4	2	352	2.28	0.352	0.80256
5	3	229	2.28	0.229	0.52212
6	4	154	2.28	0.154	0.35112
7	5	99	2.28	0.099	0.22572
8	6	87	2.28	0.087	0.19836

In Excel

Why F$10?
Plickers...

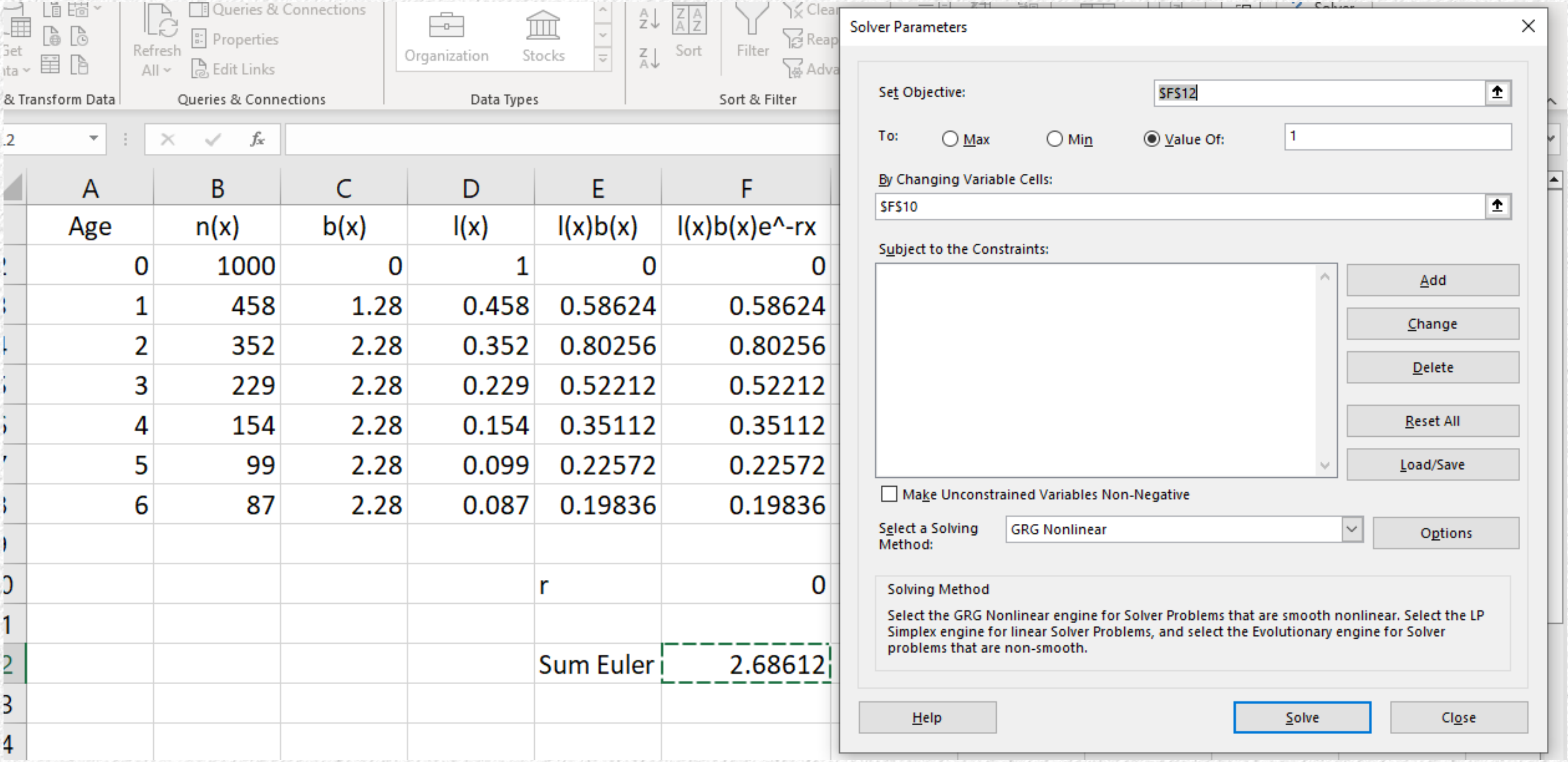
F8 =E8*EXP(-F$10*A8)

	A	B	C	D	E	F	G
1	Age	n(x)	b(x)	l(x)	l(x)b(x)	Euler	
2	0	1000	0	1	0	0	
3	1	458	1.28	0.458	0.58624	0.58624	
4	2	352	2.28	0.352	0.80256	0.80256	
5	3	229	2.28	0.229	0.52212	0.52212	
6	4	154	2.28	0.154	0.35112	0.35112	
7	5	99	2.28	0.099	0.22572	0.22572	
8	6	87	2.28	0.087	0.19836	0.19836	
9							
10					r	0	
11							
12					Sum Euler	2.68612	
13							

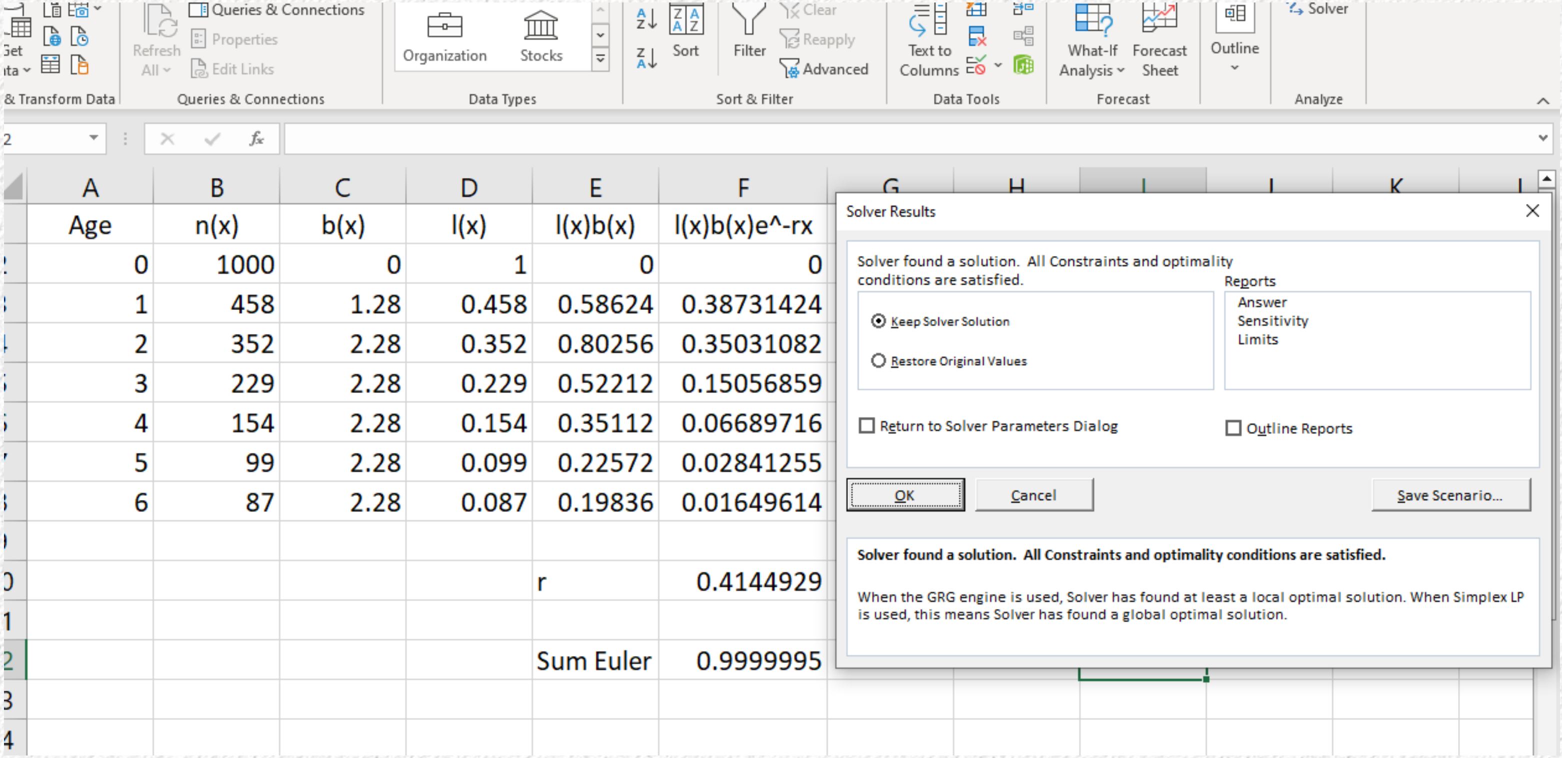
1. Solver will change this...

2. ...until this is equal to 1

Solver setup



Solution



Age	n(x)	b(x)	l(x)	l(x)b(x)	l(x)b(x)e^-rx
0	1000	0	1	0	0
1	458	1.28	0.458	0.58624	0.38731424
2	352	2.28	0.352	0.80256	0.35031082
3	229	2.28	0.229	0.52212	0.15056859
4	154	2.28	0.154	0.35112	0.06689716
5	99	2.28	0.099	0.22572	0.02841255
6	87	2.28	0.087	0.19836	0.01649614
				r	0.4144929
				Sum Euler	0.9999995

r = 0.414 is the growth rate

What if... analysis

- We could ask, how low does adult birth rate have to go for the population to stop growing?
- As population size increases females can't get enough food to reproduce successfully
- Assuming survival doesn't change, we can estimate what the reproductive rate would be when the population growth is zero

Set up in Excel

1. Set these to all point to cell c10

C4 =C$10

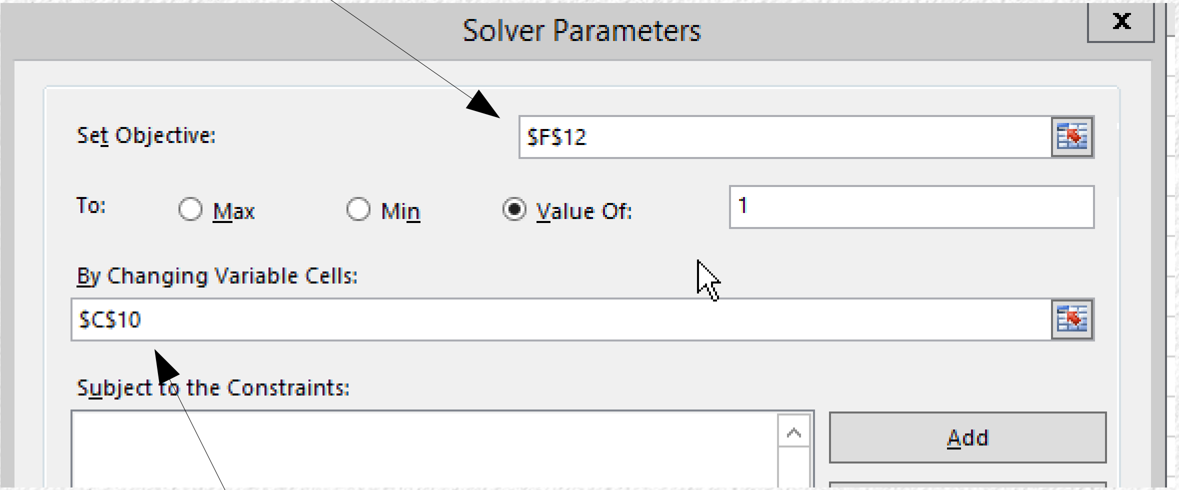
	A	B	C	D	E	F
1	**Age**	**n(x)**	**b(x)**	**lx**	**lxbx**	**Euler**
2	0	1000	0	1	0	0
3	1	458	1.28	0.458	0.58624	0.58624
4	2	352	2.28	0.352	0.80256	0.80256
5	3	229	2.28	0.229	0.52212	0.52212
6	4	154	2.28	0.154	0.35112	0.35112
7	5	99	2.28	0.099	0.22572	0.22572
8	6	87	2.28	0.087	0.19836	0.19836
9						
10		Adult birth rate	2.28		r	0
11						
12					Sum Euler	2.68612
13						
14						

2. Varying this will now change all the adult birth rates

3. Set r to 0, don't vary it

Like before, set the sum of Euler's equation to 1

Solver setup



But, now change adult birth rate instead of growth rate

C4 =C$10

	A	B	C	D	E	F
1	**Age**	**n(x)**	**b(x)**	**lx**	**lxbx**	**Euler**
2	0	1000	0	1	0	0
3	1	458	1.28	0.458	0.58624	0.58624
4	2	352	2.28	0.352	0.80256	0.80256
5	3	229	2.28	0.229	0.52212	0.52212
6	4	154	2.28	0.154	0.35112	0.35112
7	5	99	2.28	0.099	0.22572	0.22572
8	6	87	2.28	0.087	0.19836	0.19836
9						
10		Adult birth rate	2.28		r	0
11						
12					Sum Euler	2.68612
13						
14						

Solver's solution

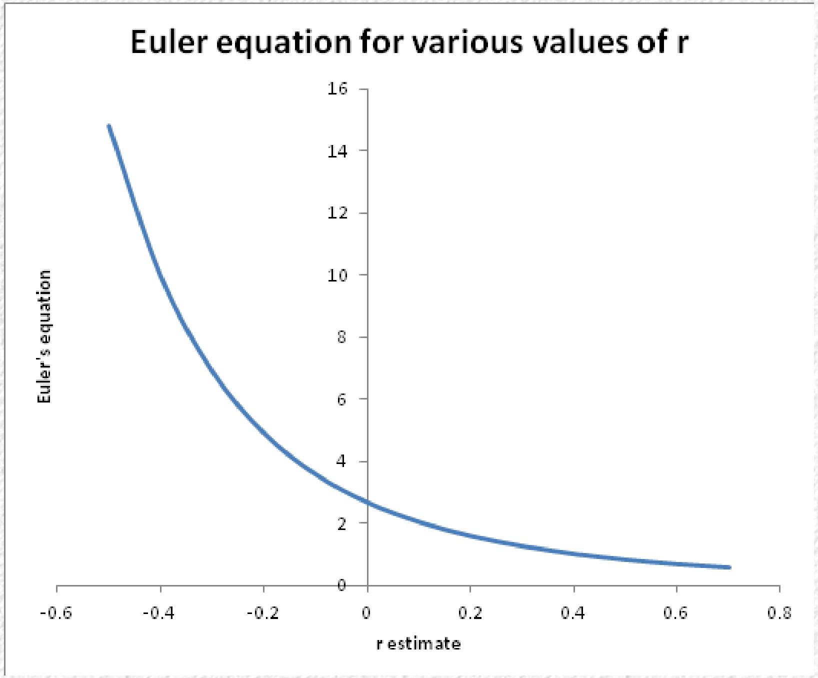
F12 | f_x | =SUM(F2:F8)

	A	B	C	D	E	F
1	**Age**	**n(x)**	**b(x)**	**lx**	**lxbx**	**Euler**
2	0	1000	0	1	0	0
3	1	458	1.28	0.458	0.58624	0.58624
4	2	352	0.4492	0.352	0.1581359	0.1581359
5	3	229	0.4492	0.229	0.10287819	0.10287819
6	4	154	0.4492	0.154	0.06918446	0.06918446
7	5	99	0.4492	0.099	0.04447572	0.04447572
8	6	87	0.4492	0.087	0.03908473	0.03908473
9						
10	Adult birth rate		0.4492		r	0
11						
12					Sum Euler	0.999999
13						
14						

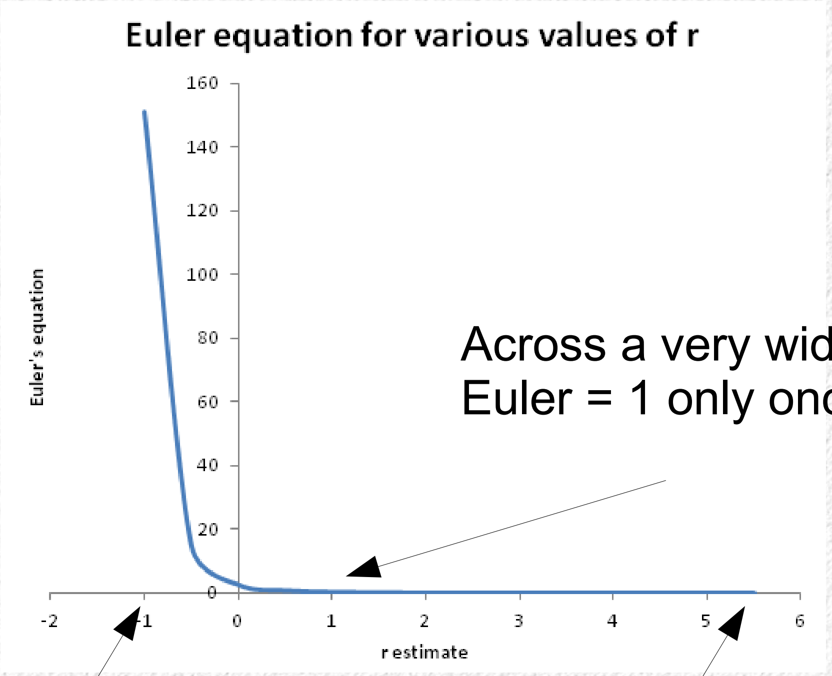
Birth rate would need to be 0.4492 for the population to stop growing

How do we know there is only one solution for r?

- For a numerical result, can't know for sure
- Some ways to check
  - Graph the result across all plausible values of r
    - Really huge r would be hard to miss biologically
  - Try different starting values in Solver to see if the solution is always the same



Wider range of possible r's



Across a very wide range,
Euler = 1 only once

40% as many next year as this year

245 times as many next year as this year