#### Numerical approaches to analysis

#### Using numerical solutions to problems The Solver in Excel

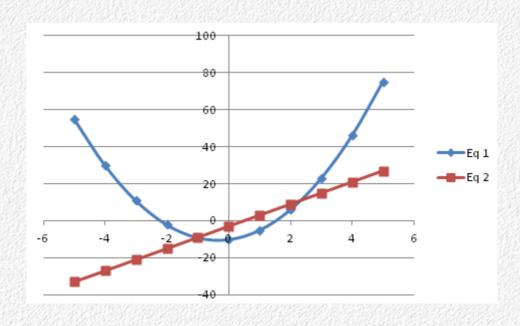
## Analytical vs. numerical solutions

- Analytical solutions = solving for the term of interest in an equation
- Numerical solutions = approximate solutions found with iterative, trial and error methods
- Numerical solutions are approximate, but if the approximation is good enough it doesn't matter
- Example: solutions to a system of equations

## Solutions to equations

- Two equations, one linear and one quadratic
  - $-y = 3x^2 + 2x 10$
  - -y = 6x 3
- Roots of the equations: where do the lines cross?
  - That is, what values of x have the same y for both equations?
- First, graph them how many solutions should we expect?
- There is an analytical solution to this

How many solutions places do the line cross? Plickers...



## The analytical solution

$$y=3x^{2}+2x-10$$
  
 $y=6x-3$ 

$$6x - 3 = 3x^2 + 2x - 10$$

 $0=3 x^{2}-4 x-7$ Quadratic equation in standard form, find roots with quadratic formula  $\rightarrow$ 

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{-(-4)+\sqrt{(-4)^2-4(3)(-7)}}{2(3)}$$

$$x = 2\frac{1}{3}, \quad y = 11$$
Plug x into  
either equation  
to get y
$$x = -1, \quad y = -9$$

$$\frac{-(-4)-\sqrt{(-4)^2-4(3)(-7)}}{2(3)}$$

#### Numerical approach

• Re-write the equations as:  $0=3x^2+2x-10-y$ 

$$0 = 6 x - 3 - y$$

- Set values for X and Y to be the same in both equations
  - Initial guesses will make the equations not equal 0 at first
  - Change the values of X and Y until both equations equal 0
- We'll get solutions for both X and Y at the same time

#### Numerical method – in Excel

	Α	В	С	
1	F1	- 3x^2+2x -1	.0 - y	The initial
2	F2	6x - 3 - y		setup
3				
4	Х	0		1. Enter initial guesses for x
5	у	0		and y
6				
7	F1	-10		2. Calculate F1 and F2
8	F2	-3		using x and y
9				-3. Square F1
10	Sum sq	109	4	and F2 and sum them
11			1 15 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1	

## Try new set of numbers for x and y

6633	1889 - 1996 - 1996 - 1996 - 1996 - 1996 - 1996 - 1996 - 1996 - 1996 - 1996 - 1996 - 1996 - 1996 - 1996 - 1996 - 1997 - 1997 - 1996 - 1996 - 1996 - 1996 - 1996 - 1996 - 1996 - 1996 - 1996 - 1996 - 1996 - 1996 - 1996 - 1996 -	723382833838522322233		
	A	В	С	
1	F1	3x^2+2x -1	.0-у	
2	F2	6x - 3 - y		
3				1. Change
4	Х	1		the values of x and y
5	у	1		
6				2. Calculate F1 and
7	F1	-6		F2 using new values
8	F2	2		for x and y
9				3. Sum of squared
10	Sum sq	40		values gets closer to (
11				<ul> <li>moving in the right direction!</li> </ul>

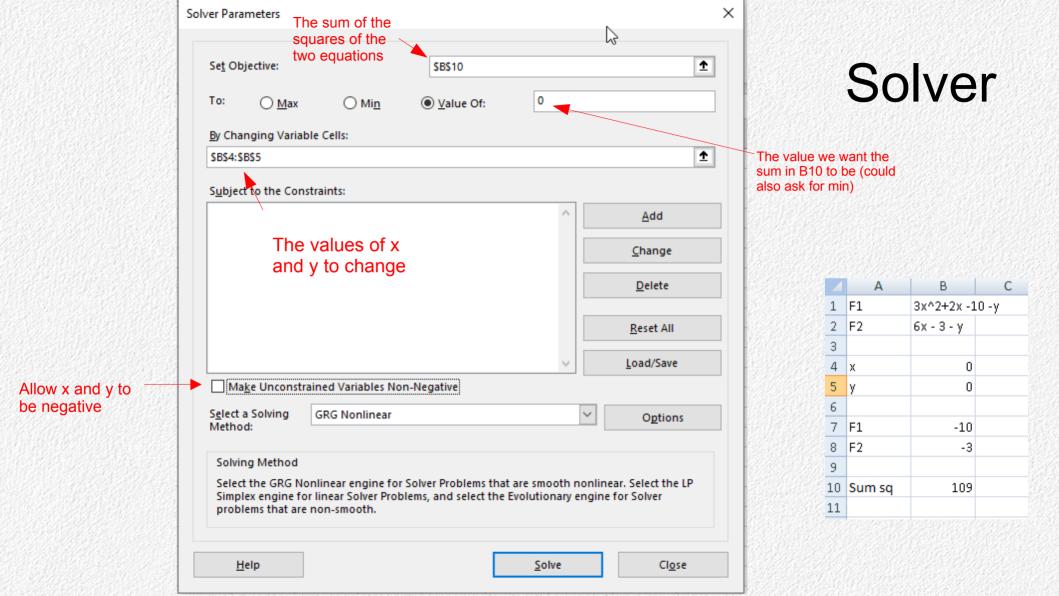
## Not 0 yet...

- Better
- Blind search could take a really long time
- There are good search algorithms that converge on solutions quickly
- Excel's Solver uses these

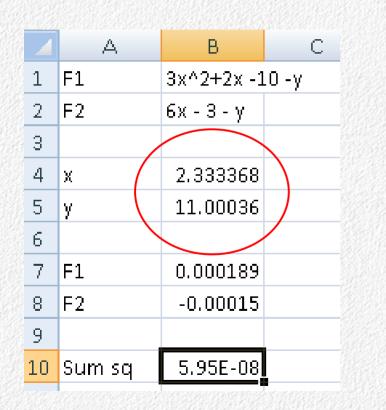
	А	В	С
1	F1	3x^2+2x -1	.0 - y
2	F2	6x - 3 - y	
3			
4	х	2	
5	у	10	
6			
7	F1	-4	
8	F2	-1	
9			
10	Sum sq	17	
11			

# Optimization algorithms used by Excel's Solver

- Excel picks a method to use based on the formulas in the spreadsheet
  - For linear problems, uses the Simplex method
  - For non-linear problems it uses a generalized gradient method
- Both require initial guesses of the solutions
- Both can accept constraints (i.e. only positive values considered)
- Both are iterative (i.e. new values chosen until no more improvement at the level of precision desired)



#### The first solution

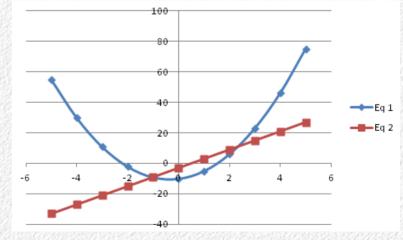


Solve	er Results
Solver found a solution. All Constraints and op conditions are satisfied.	Re <u>p</u> orts
<u>Keep Solver Solution</u> Restore Original Values	Answer Sensitivity Limits
Return to Solver Parameters Dialog	☐ O <u>u</u> tline Reports
<u>Q</u> K <u>C</u> ancel	Save Scenario
Solver found a solution. All Constraints and opt	imality conditions are satisfied.
When the GRG engine is used, Solver has found is used, this means Solver has found a global o	d at least a local optimal solution. 🏹 hen Simplex optimal solution.

 $x=2\frac{1}{3}, y=11 \leftarrow Do they match?$ Plickers...

### What about the second solution?

- There are two points of intersection of the lines  $\rightarrow$  two solutions
- Our trial and error approach gave us one, now we need
   the other
- To get the second, try another starting point closer to the other solution and run Solver again



#### Start closer to the second solution...

	А	В	С
1	F1	3x^2+2x -1	.0 -y
2	F2	6x - 3 - y	
3			
4	Х	-1	
5	у	-10	
6			
7	F1	1	
8	F2	1	
9			
10	Sum sq	2	
11	17587973989759977479974334	2-12-5-17-17-17-17-17-17-17-17-17-17-17-17-17-	

#### Second solution

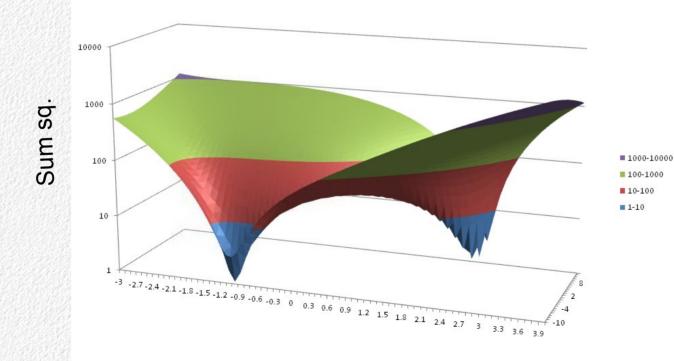
	Α	В	С
1	F1	3x^2+2x -1	_
2	F2	6х-З-у	•
3			
4	х /	-1	
5	у	-8.99996	
6			
7	F1	-5.6E-05	
8	F2	-1.4E-05	
9			
10	Sum sq	3.32E-09	
11			

Note that this works best when you know how many solutions to expect – graphing is a very important first step

If you don't know, choose several different starting positions to increase the chance you'll find the solutions

$$x = -1, y = -9$$

## Sum of squared functional values at different x,y values



The slope of the surface determines which solution you'll find

Usually, whichever you start closer to is the one you'll find

## Numerical methods in biology

- Some equations can't be solved analytically, have to use numeric methods
- Example: life tables
  - Data on age-specific birth and death rates for populations
- The basic data are:
  - The number of individuals alive each year, starting from birth until all are dead =  $n_x$
  - The number of female offspring per females of age x is  $b_x$

### Life table for a squirrel population

	А	В	С
1	Age	n(x)	b(x)
2	0	1000	0
3	1	458	1.28
4	2	352	2.28
5	3	229	2.28
6	4	154	2.28
7	5	99	2.28
8	6	87	2.28

#### **Euler's equation**

- Population growth rate is the balance between birth and death rate, r = birth rate – death rate
- If r is positive, the population is growing
- The best estimate of r from a life table is the value that satisfies Euler's equation:

$$1 = \sum l_x b_x e^{-rx}$$

- x (age),  $I_x$ ,  $b_x$  are all known, e is a constant
- Equation can't be solved analytically, but we can find r numerically with the Solver

# Convert number alive $(n_x)$ to proportion alive $(I_x)$

	А	В	С	D
1	Age	n(x)	b(x)	l(x)
2	0	1000	0	1
3	1	458	1.28	0.458
4	2	352	2.28	0.352
5	3	229	2.28	0.229
6	4	154	2.28	0.154
7	5	99	2.28	0.099
8	6	87	2.28	0.087
				5754538557756775327387853538

# Multiply proportion alive by birth rate $(I_x b_x)$

	А	В	С	D	E
1	Age	n(x)	b(x)	l(x)	l(x)b(x)
2	0	1000	0	1	0
3	1	458	1.28	0.458	0.58624
4	2	352	2.28	0.352	0.80256
5	3	229	2.28	0.229	0.52212
6	4	154	2.28	0.154	0.35112
7	5	99	2.28	0.099	0.22572
8	6	87	2.28	0.087	0.19836

#### In Excel

Why F\$10? Plickers...

28/20	22.14.18.22	1999/100	ちちちちろんちむ	1945 6 6 6 6 1				T IICKC/3
		F8	•	- (	<i>f</i> <sub>*</sub> =E8*1	EXP(-F\$10*4	48)	
	Α	В	С	D	E	F	G	
1	Age	n(x)	b(x)	I(x)	l(x)b(x)	Euler		
2	0	1000	0	1	0	0		
3	1	458	1.28	0.458	0.58624	0.58624		
4	2	352	2.28	0.352	0.80256	0.80256	and the second se	
5	3	229	2.28	0.229	0.52212	0.52212		
6	4	154	2.28	0.154	0.35112	0.35112		
7	5	99	2.28	0.099	0.22572	0.22572		
8	6	87	2.28	0.087	0.19836	0.19836		
9								
10					r	0		
11								1. Solver will
12					Sum Euler	2.68612		change this
13						1		onungo ano

#### Solver setup

Get Ita ~		Generation for the second seco	Connections			$ \begin{array}{c c} \begin{array}{c} \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\$
& Tra	ansform Data	Queries & Conn	ections	Data Type	s	Sort & Filter
.2	• :	$\times \checkmark f_x$				
	А	В	С	D	E	F
	Age	n(x)	b(x)	l(x)	l(x)b(x)	l(x)b(x)e^-rx
2	0	1000	0	1		0 0
ģ.	1	458	1.28	0.458	0.5862	0.58624
ł	2	352	2.28	0.352	0.8025	0.80256
3	3	229	2.28	0.229	0.5221	0.52212
3	4	154	2.28	0.154	0.3511	0.35112
'	5	99	2.28	0.099	0.2257	0.22572
3	6	87	2.28	0.087	0.1983	0.19836
)						
C					r	0
1						
2					Sum Eule	er 2.68612
3						
4						

			Calvar	
olver Parameters				×
Se <u>t</u> Objective:		SFS12		Ť
To: <u>M</u> ax	◯ Mi <u>n</u>	() <u>V</u> alue Of:	1	
<u>B</u> y Changing Variat	ole Cells:			
SFS10				Î
Subject to the Con	straints:			
			^	<u>A</u> dd
				<u>C</u> hange
				<u>D</u> elete
				<u>R</u> eset All
			~	Load/Save
Ma <u>k</u> e Unconstr	ained Variables No	on-Negative		
S <u>e</u> lect a Solving Method:	GRG Nonlinear		~	O <u>p</u> tions
	r linear Solver Pro	r Solver Problems tha blems, and select the		
<u>H</u> elp			<u>S</u> olve	Cl <u>o</u> se

### Solution

-=== [	Re Re	C Queries & C C Properties All ~ & Edit Links		_	ocks ⊽ Z↓	Sort Filter	pply Text to T				
& Tran	sform Data	Queries & Conne	ctions	Data Types	;	Sort & Filter	Data Tools Forecast Analyze A				
2	-	$\times \checkmark f_x$					×				
	А	В	С	D	Е	F	G H I I K I				
8	Age	n(x)	b(x)	l(x)	l(x)b(x)	l(x)b(x)e^-rx	Solver Results X				
<u>.</u>	0	1000	0	1	0	0	Solver found a solution. All Constraints and optimality conditions are satisfied. Reports				
3	1	458	1.28	0.458	0.58624	0.38731424	Kep Solver Solution     Sensitivity				
).	2	352	2.28	0.352	0.80256	0.35031082	Acep solver solution     Sensitivity     Limits				
2	3	229	2.28	0.229	0.52212	0.15056859					
;	4	154	2.28	0.154	0.35112	0.06689716	Return to Solver Parameters Dialog Outline Reports				
'	5	99	2.28	0.099	0.22572	0.02841255					
3	6	87	2.28	0.087	0.19836	0.01649614	QK         Cancel           Save Scenario         Save Scenario				
4							Solver found a solution. All Constraints and optimality conditions are satisfied.				
С				r		0.4144929	When the GRG engine is used, Solver has found at least a local optimal solution. When Simplex LP				
1							is used, this means Solver has found a global optimal solution. When Simplex LP				
2					Sum Euler	0.9999995					
3											
4											

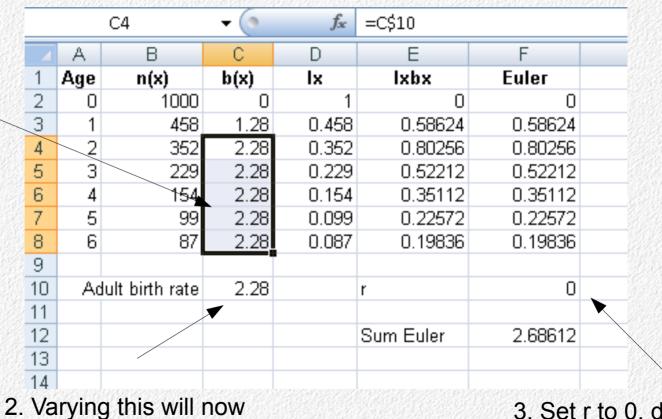
r = 0.414 is the growth rate

## What if... analysis

- We could ask, how low does adult birth rate have to go for the population to stop growing?
- As population size increases females can't get enough food to reproduce successfully
- Assuming survival doesn't change, we can estimate what the reproductive rate would be when the population growth is zero

## Set up in Excel

1. Set these to all point to cell c10



change all the adult birth rates

3. Set r to 0, don't vary it

#### Like before, set the sum of Euler's equation to 1

## Solver setup

F\$12	
1 312	<b>E</b>
Value Of: 1	
$\searrow$	
	<b>E</b>
	N

But, now change adult birth rate instead of growth rate

C4			- ( )	$f_x$	=C\$10	
4	Α	В	С	D	E	F
1	Age	n(x)	b(x)	lx	lxbx	Euler
2	0	1000	0	1	0	0
3	1	458	1.28	0.458	0.58624	0.58624
1	2	352	2.28	0.352	0.80256	0.80256
5	3	229	2.28	0.229	0.52212	0.52212
6	4	154	2.28	0.154	0.35112	0.35112
7	- 5	99	2.28	0.099	0.22572	0.22572
8	6	87	2.28	0.087	0.19836	0.19836
9						
0	Ad	ult birth rate	2.28		r	0
1						
2					Sum Euler	2.68612
3						
4						

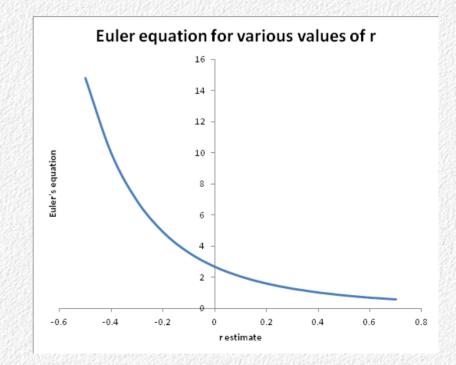
#### Solver's solution

F12			-	$f_{x}$	=SUM(F2:F8)	
4	Α	В	С	D	E	F
1	Age	n(x)	b(x)	İx	lxbx	Euler
2	0	1000	0	1	0	0
3	1	458	1.28	0.458	0.58624	0.58624
4	2	352	0.4492	0.352	0.1581359	0.1581359
5	3	229	0.4492	0.229	0.10287819	0.10287819
6	4	154	0.4492	0.154	0.06918446	0.06918446
7	5	99	0.4492	0.099	0.04447572	0.04447572
8	6	87	0.4492	0.087	0.03908473	0.03908473
9						
0	Ac	lult birth rate	0.4492		r	0
1			◀			
2					Sum Euler	0.999999
3						
Δ						

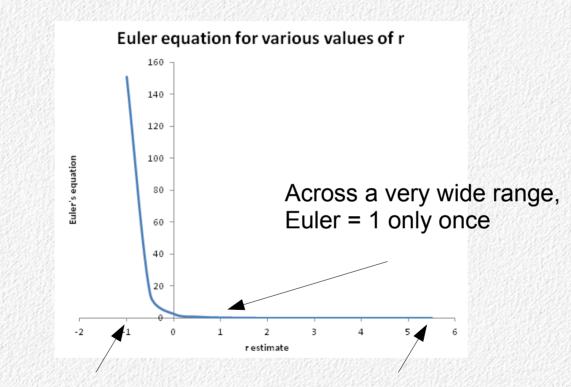
Birth rate would need to be 0.4492 for the population to stop growing

## How do we know there is only one solution for r?

- For a numerical result, can't know for sure
- Some ways to check
  - Graph the result across all plausible values of r
    - Really huge r would be hard to miss biologically
  - Try different starting values in Solver to see if the solution is always the same



## Wider range of possible r's



40% as many next year as this year

245 times as many next year as this year